Preface

It is a pleasure for me to have the opportunity to write a preface for this book on the foundations of nonstandard analysis by Vladimir Kanovei. It may seem that everything relevant has been said concerning this theme. That this is not so emerges from the last three chapters of this book which can be seen as the conclusion of an effort starting in 1977 with the introduction of Internal Set Theory by Edward Nelson. There has been an unfortunate dispute in the West concerning the adequate way to do nonstandard analysis. The dominant "religion" used to be the method of nonstandard extensions of superstructures. This is a method which is based on the idea of extending a given sufficiently big fragment of the set universe (a so called superstructure) to a bigger one which contains all those objects the founders of the calculus had been talking about. From a conceptual point of view this is attractive because it conforms to the mainstream of modern analysis. It also did away with the logical intricacies of Robinson's original approach. With the advent of Loeb measures this approach even came into possession of a valuable pawn guaranteeing that no approach unable to incorporate these measures in a natural way could claim to be an adequate substitute for the superstructure approach.

The alternative proposed in 1977 by Edward Nelson quickly found isolated enthusiasts sometimes praising it in highest terms. Axiomatizing an internal universe (therefore its name IST: Internal Set Theory) it was unable to work with external sets in an intrinsic manner so that the Loeb construction seemed out of reach. In the superstructure approach there is no problem with that because the whole nonstandard superstructure is embedded in the conventional set universe. External predicates in the nonstandard superstructure will in general not define internal sets within the nonstandard superstructure but will define plain sets in the set universe. The construction of the Loeb measure cleverly plays upon this fact. But that merit also is connected to the philosophical defects of the superstructure approach. The final result of such a free use of external sets leads to interesting mathematical objects but they are not part of the internal superstructure. Their construction uses properties of the internal world of the nonstandard superstructure but after completion of the construction the internal world is abandoned. It is always awkward if results are derived by an elaborate construction all the traces of which are eliminated in the final result. It is more than an aesthetic whim to look for the conceptual unity of a mathematical result and its proof.

This unsatisfactory state of affairs was also felt by several logicians. It seems that Kreisel was the first to ask if nonstandard analysis could not be given an intrinsic axiomatic formulation independent of model theoretic constructions. There have been several attempts by logicians to formulate axiomatic theories which would codify the existing practice of nonstandard analysis. There are three theories proposed by Hrbaček and also some by Kawaï. They all incorporate external sets into their universe.

In contrast with this Nelson wanted to develop an extended language for conventional mathematics which would make certain intrinsic intuitions available for the mathematician. Although it is a fruitless exercise to speculate on the intentions and ideas of Leibniz with respect to infinitesimals there is a letter in which he explains to one of his followers infinitesimals to be a *facon de parler*, a way to shorten the convoluted arguments of Archimedes concerning exhaustion. In modern language this sounds as if he wanted to say that infinitesimals are a means of reformulating epsilontics in a way more adequate for our intuition. In modern times the idea that nonstandard analysis could be considered not as an extension introducing new objects but rather as an extension of the language which provides us with new deductive procedures goes actually back to Robinson himself.

As mentioned before, there was a dispute in the West between the “asterisk people” ornating their nonstandard extensions by an asterisk) and the aficionados of IST (denoting their enriched standard sets by the traditional symbols). This dispute occasionally took somewhat bizarre forms. Mathematicians in the former Soviet Union had a much more detached and fruitful attitude towards this dichotomy. They would use freely both approaches whichever would better suit their respective problems. This book is a good example of this reasonable attitude. In the first two chapters the superstructure approach is developed along the traditional lines. In the next three chapters a systematic study of internal theories is undertaken. The two main theories presented are IST (Nelson's Internal Set Theory) and BST (Bounded Set Theory as introduced in this book). The exposition is such that the relation between the superstructure approach and these internal theories is always visible. Nelson's construction of adequate ultralimits is streamlined quite a bit. And the result is just most pleasing: it is shown in the last chapter that BST is precisely what Nelson wanted IST to be, a *reformulation of conventional mathematics*. It is proved that all theorems of BST are *effectively reducible* to theorems of ZFC and all the theorems of the latter theory are theorems of BST. The author also shows that Nelson's IST proper does contain a sentence which is not reducible to a sentence of ZFC.

Nelson wanted IST to have this property of reducibility but he proved it only for IST enriched by the language of a model of IST. But in his paper in the Ann. Pure and Appl. Log. 38 he gives a very clear account of his principal idea. He uses the example of "fixed". Consider the statement: for all fixed positive numbers x from R there is a smaller positive number y. This is a true statement about R independent of the meaning of "fixed". What about the statement "there is a positive number y which is smaller than any fixed number x"? To decide this question we should know what "fixed" means. But we could also turn things upside down and say "fixed" expresses not a property but is a syntactic device defined to make the two statements equivalent *by definition*. To say that there exists a positive number which is smaller than any fixed number is then only an unorthodox convention to express the first statement in a way in which quantifiers have changed place.

More precisely, we augment not the universe but only the language of ZFC by symbols which are called external quantifiers and give rules of manipulation for these. These rules are such that any closed statement of the new theory can be transformed into a statement of ZFC which is provable in ZFC if and only if the original statement is provable in the new theory. BST which is treated in detail in chapter V of this book is precisely this theory. IST proper is not of this kind and Nelson's theory IST enriched by the language of a model of IST is unsatisfactory from a philosophical point of view.

This book is by no means restricted to the study of BST. In chapters III and IV IST is studied and some interesting facts about it are proved. These sections need a certain familiarity with set theory and foundations of mathematics. Although not familiar with these disciplines myself I think mathematicians from these fields will find these parts of the book interesting. They are written well enough that even a non-expert like myself can gain some insight into the working of this internal theory and why it may provide interesting counterexamples making it a useful tool for the set theorist.

But any internal theory suffers from the defect that a construction like the Loeb measure seems to be beyond the reach of any such theory. The reason is that the universe consists of internal sets only. Even on a more modest level the lack of "sets" defined by external formulas is felt. Many authors have used a semi-intuitive notion of external set in IST. The author of this preface has introduced a kind of local parametrization of an external set described by an arbitrary external formula. The idea behind is to introduce quantifiers over a specific type of "external sets" through the backdoor by parametrizing these by internal sets such that quantification is actually expressed in terms of the internal sets. Kanovei discovered that BST has the property that all bounded classes of BST (a bounded class being the collection of elements contained in a set and satisfying an arbitrary external predicate) can be described by a single fixed external formula and a suitable internal set as parameter in this formula. This is a consequence of the unrestricted working of Nelson's reduction algorithm in BST.

This allows to formally introduce an external universe (of bounded classes) and a first order language on it with a translation algorithm translating any sentence in this language to a sentence in the language of BST. The truth of a sentence in the extended universe is expressed by the truth of the translated sentence in BST. This external universe of bounded classes can be extended to one which contains classes of classes and so on. The author then studies the formal properties of these external universes. They do not admit the formation of power sets which is an essential step in the Loeb construction. But he also shows that there is a restricted external universe in which power set formation is possible such that the Loeb construction can be successfully tackled with. The approach is different from the external theories of Hrbaček and Kawaï in so far as the external universe over BST and its language can be seen as a syntactic reformulation of BST itself. Thus, the external language can again be interpreted as a new language to express statements of ZFC because any theorem in the external language is a theorem of BST (by construction) which then can be reduced to a theorem of ZFC. In the age of computers this brings immediately an analogy to the mind: the language of BST and the language of the extended universe are related to the language of ZFC like high level programming languages to the underlying machine language. The content of the high level languages is expressible in the machine language but a lot of very intelligible sentences of a high level language become unintelligible in their translation to machine language.

To recapitulate what has been said before, there are two different interpretations of what BST describes:

- the orthodox interpretation sees BST as the description of a set universe which contains the set universe of ZFC in the sense that the subuniverse of standard sets of BST satisfies all the axioms of ZFC. But all the standard infinite sets of BST contain nonstandard elements. Thus the language of BST describes elements in these sets the language of ZFC cannot describe. One could therefore imagine that the language of BST makes "visible" elements in the sets of ZFC which remain "invisible" in ZFC because of the reduced language.

- the unorthodox interpretation sees BST as the description of the set universe of ZFC by means of an extended language. The external quantifiers have no semantic meaning, they are elements of the extended language the use of which is determined by additional axioms in such a way that any sentence of BST can be transformed by Nelson's algorithm into a sentence of ZFC which is a theorem of ZFC if and only if the original sentence is a theorem of BST. This interpretation rests on the fact that no nonstandard set of BST can be defined uniquely by a parameter-free formula or one containing only standard parameters. All these nonstandard sets in the orthodox interpretation making this bigger world so interesting are in this sense mere fictions. We may thus deny their existence totally and see BST as the description of the universe of ZFC in a more powerful language without addition of any new content.

Which of the last mentioned two interpretations is preferred is largely a question of taste. But Nelson's interpretation has in our view some definite advantages. One of them is that in this interpretation no new quantities beyond those we have always been talking about are introduced into mathematical discourse (as was mentioned before this idea goes back to Robinson). The second advantage is that the predicate "standard" interpreted in this way mimicks a construct which is present in natural languages but which has been banned from conventional mathematical language. Natural languages express two different kinds of properties: properties which could be called *quantitative* and others which should be called *qualitative.* The first mentioned are adequately modeled by conventional mathematics and are of the kind that they can be expressed by formulas in the language of ZFC which means they can be reduced to the property of being a member of a specific set or not. The last mentioned are not of this kind, in natural language they are usually explicitly or implicitly modified by adverbs like "approximately", "nearly" and others of this kind or are implicitly of a relative character like "large" or "long".

To say $x > b$ is a quantitative statement about the real number x: it means that x is in the set of numbers *larger* than b. But what does it mean for x to be *large*? It does not make sense to *define* x is *large* if $x > b$ for a suitable b because we automatically assume that if x is large then $x-1$ will be large. This is wrong if we define large in the way just proposed. Even more can be said: whatever definition in the language of ZFC we may choose it will defeat that intuitive property of large numbers. If we add the unary predicate st and the axioms governing its use then it turns out that the nonstandard numbers in R satisfy the intuitive properties of large numbers. The standard numbers are then numbers of moderate size (another qualitative notion which cannot be given a sense in ZFC respecting the intuition related to it).

Analyzing the use people in physics and other natural sciences are making of the notions “large”, “small” and others of a similar kind one realizes that they are always used in a relative sense: the number is large (or small) with respect to certain parameters in the context considered. *BST is the miraculous universal formalization of such context dependent notions to all of mathematics without specifying the context at all.* Starting from N one can infer intuitive interpretations for “standard” in Z, Q, R and C and other conventional sets. But it should be understood that in this interpretation there are no new numbers or mathematical objects beyond those of conventional set theory. They only have gained something like a relative quality which may be expressed in a (often quite convoluted) way by quantitative properties of ZFC.

When thinking of nonstandard mathematics one has, I think, to make a decision. Either nonstandard mathematics is a jargon to rationalize the technical construction of ultrapowers and ultralimits which lead to mathematical objects of an interesting nature or one sees it as the expression of a fundamental intrinsic property of conventional set theory. BST is the theory which describes precisely what this intrinsic property of ZFC is: it is the possibility to extend the language by a new type of predicate such that in the extended language statements may become syntactically simpler than in the original language. BST is a theory which does not produce “new” theorems in the sense that they could not be reduced to theorems of ZFC. But in the extended language they are new, they offer new insights and they often throw a highly valuable light on conventional concepts of a very technical nature. Its reducibility to ZFC leaves no room for philosophical disputes about the admissibility of such a theory. In contrast to this one of the external theories of Hrbaček is stronger than ZFC while about the reducibility of his other external theories and those of Kawaï nothing seems to be known. But I doubt that any one of these theories will be reducible to ZFC. And as long as there is a whole range of different external theories none of which is presenting an intrinsic reason why it should be preferred over the others BST is the canonical choice.

Wuppertal, May 1993 Michael Reeken